

A Family of Counter Examples to an Approach to Graph Isomorphism

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Abstract

We give a family of counter examples showing that the two sequences of polytopes $\Phi_{n,n}$ and $\Psi_{n,n}$ are different. These polytopes were defined recently by S. Friedland in an attempt at a polynomial time algorithm for graph isomorphism.

1 Introduction

In a recent posting at arXiv (arXiv:0801.0398v1 [cs.CC] 2 Jan 2008 and arXiv:0801.0398v2 [cs.CC] 4 Jan 2008), S. Friedland defined two sequences of polytopes $\Phi_{n,n}$ and $\Psi_{n,n}$.

Let $\Omega_n \subset \mathbf{R}_+^{n \times n}$ denote the $n \times n$ doubly stochastic matrices. Then $\Psi_{n,n} \subset \Omega_{n^2}$ is the convex hull of the tensor products $A \otimes B$, where $A, B \in \Omega_n$. Meanwhile $\Phi_{n,n}$ is defined to be the subset of Ω_{n^2} defined by the following set of linear constraints.

$$\sum_{j,l=1}^{n,n} c_{(i,k),(j,l)} = \sum_{j,l=1}^{n,n} c_{(j,l),(i,k)} = 1, i = 1, \dots, n, k = 1, \dots, n,$$

$$\sum_{j=1}^n c_{(i,k),(j,l)} = \sum_{j=1}^n c_{(1,k),(j,l)}, \sum_{j=1}^n c_{(j,k),(i,l)} = \sum_{j=1}^n c_{(1,k),(j,l)},$$

where $i = 2, \dots, n$, and $k, l = 1, \dots, n$,

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$$\sum_{l=1}^n c_{(i,k),(j,l)} = \sum_{l=1}^n c_{(i,1),(j,l)}, \sum_{l=1}^n c_{(i,l),(j,k)} = \sum_{l=1}^n c_{(i,1),(j,l)},$$

where $i = 2, \dots, n$, and $k, l = 1, \dots, n$.

It was shown that $\Psi_{n,n} \subseteq \Phi_{n,n}$. (In the earlier version it was claimed that $\Psi_{n,n} = \Phi_{n,n}$. If this were the case, then graph isomorphism would be in P, as one can reduce the problem to linear programming. In the Jan 4th version Friedland stated that the equality $\Psi_{n,n} = \Phi_{n,n}$ “is probably wrong”.) In this note we give an explicit family of counter examples showing $\Psi_{n,n} \neq \Phi_{n,n}$. For every $n \geq 4$, our examples consist of an exponential number of matrices which are vertices of $\Phi_{n,n}$, but do not belong to $\Psi_{n,n}$.

2 Counter Examples

Let $\rho \in S_n$ be the cyclic permutation $(1 \ 2 \ 3 \ \dots \ n)$. Let $\sigma \in S_n$ be any permutation.

Lemma 2.1. *There are exactly $n! - n\phi(n)$ many permutations $\sigma \in S_n$, such that $\sigma\rho\sigma^{-1}$ does not belong to the subgroup generated by ρ .*

Proof. A conjugate $\sigma\rho\sigma^{-1}$ of ρ is also an n -cycle. To be in the subgroup generated by ρ , iff it is a power ρ^i for some i relatively prime to n . To be of this form, iff σ is of the form $\sigma(i+1) - \sigma(i)$ (in a cyclic sense) is a constant relatively prime to n , which means there are exactly $n\phi(n)$ many. \square

Let A be the matrix whose first row is (x_1, x_2, \dots, x_n) , and its i -th row is obtained by applying $(i-1)$ times the cyclic permutation ρ . Let B be the matrix whose first row is (x_1, x_2, \dots, x_n) permuted by σ , and its i -th row is obtained by further applying $(i-1)$ times the cyclic permutation ρ .

Lemma 2.2. *Whenever $\sigma \in S_n$ satisfies Lemma 1, there does not exist a pair of permutation matrices P and Q , such that $A = PBQ$.*

Proof. The first two rows of B are $\sigma(x_1, x_2, \dots, x_n)$ and $\rho\sigma(x_1, x_2, \dots, x_n)$. Assume for contradiction that there does exist a pair of permutation matrices P and Q , such that $A = PBQ$. The first two rows of BQ are $q\sigma(x_1, x_2, \dots, x_n)$ and $q\rho\sigma(x_1, x_2, \dots, x_n)$, where q is the permutation corresponding to Q . They must be two rows of A , so there exist i and j ($i \neq j$) such that $q\sigma(x_1, x_2, \dots, x_n) = \rho^i(x_1, x_2, \dots, x_n)$ and $q\rho\sigma(x_1, x_2, \dots, x_n) = \rho^j(x_1, x_2, \dots, x_n)$. We get $\sigma^{-1}\rho\sigma = \rho^{j-i}$, contradicting with lemma 1. \square

Suppose $A = (a_{ij})$ is an $n \times n$ matrix. we use \hat{A} to denotes the column vector $(a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, a_{31}, \dots, a_{nn})^T$ of length n^2 .

Given A and B , define T to be the $n^2 \times n^2$ matrix composed of 0 and $1/n$ such that $\hat{A} = T\hat{B}$.

An example of this is shown as follows, for $n = 4$ and $\sigma = (3\ 4)$:

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_1 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & x_1 & x_2 & x_3 \end{pmatrix}, B = \begin{pmatrix} x_1 & x_2 & x_4 & x_3 \\ x_2 & x_4 & x_3 & x_1 \\ x_4 & x_3 & x_1 & x_2 \\ x_3 & x_1 & x_2 & x_4 \end{pmatrix},$$

$$T = 1/4 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Theorem 2.1. *For any $\sigma \in S_n$ satisfying Lemma 1.1, the matrix T is an extreme point of $\Phi_{n,n}$. However, $T \notin \Psi_{n,n}$.*

Proof. By the definition of A , B and $T = (t_{(i,k),(j,l)})$, for each fixed pair i, j , $(t_{(i,k),(j,l)})$ (respectively, for each fixed k, l , $(t_{(i,k),(j,l)})$) is a permutation matrix multiplied by $1/n$. Obviously, $T \in \Phi_{n,n}$. For each double row index (i, k) , either fix i , or fix k , and varying the other index, and for each double column index (j, l) , either fix j , or fix l , and varying the other index, we always get an n by n permutation matrix.

Suppose $T = \sum_s w_s T_s$, where $T_s \in \Phi_{n,n}$, $w_s > 0$, and $\sum_s w_s = 1$. So within each block (fixed i, j , varying k and l ,) the non-zero entries of T_s are a subset of non-zero entries of T within that block, which form a permutation matrix. then by the equations for T_s within the block, it must be either

totally zero or a positive multiple of the same permutation matrix made up of non-zero entries of T within that block. For each block, the permutation matrix is the same for every T_s . The multipliers form a doubly stochastic matrix $M_s \in \Omega_n$, by the global sum $\sum_{j,l=1}^{n,n} = 1$. Therefore T_s is as follows: its (i, j) block is obtained by multiplying each entry of a doubly stochastic matrix $M_s \in \Omega_n$ with the permutation matrix of T for each block.

Now if we consider the sum $\sum_{j=1}^n c_{(i,k),(j,l)} = \sum_{j=1}^n c_{(1,k),(j,l)}$, by the property of T each row of M_s is a constant. (Similarly each column of M_s is a constant.) Thus M_s is just the all $1/n$ matrix $1/nJ$.

This implies that there is exactly one term in the sum $T = \sum_s w_s T_s$, and T is an extreme point.

Assume for a contradiction that $T \in \Psi_{n,n}$ and $T = \sum_s w_s P_s \otimes Q_s$, where P_s, Q_s are permutation matrices, $w_s > 0$, and $\sum_s w_s = 1$. We get $T \geq w_1 P_1 \otimes Q_1$ (Here the relation of \geq is entry-wise). For any $x_1, x_2, \dots, x_n \geq 0$, $T\hat{B} \geq w_1 P_1 \otimes Q_1 \hat{B}$, that is, $A \geq w_1 P_1 B Q_1$. By lemma 1.2, $P_1 B Q_1$ is different from A , so there must be an entry (i, j) such that they are different at that entry. Notice that each entry of A or $P_1 B Q_1$ is a single variable from $\{x_1, \dots, x_n\}$. W.l.o.g, we can assume the (i, j) -th entry of A and $P_1 B Q_1$ are x_1 and x_2 . We can set $x_1 = 0$ and $x_2 = 1$ such that $A_{ij} < (w_1 P_1 B Q_1)_{ij}$, which is a contradiction. So $T \notin \Psi_{n,n}$. \square

Before we posted this note, we note that Babai (<http://people.cs.uchicago.edu~laci/polytope.pdf>) and Onn (arXiv:0801.1410) have both pointed out that the linear optimization problem over the polytope $\Psi_{n,n}$ can solve NP-complete problems, and therefore it is unlikely that $\Psi_{n,n}$ can be defined by a polynomial number of (in)equalities as $\Phi_{n,n}$ can. In (<http://people.cs.uchicago.edu~laci/polytope-correspondence.pdf>), Babai also mention that Joel Rosenberg already gave a counter example showing the two polytopes are different, for $n = 4$.

References

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